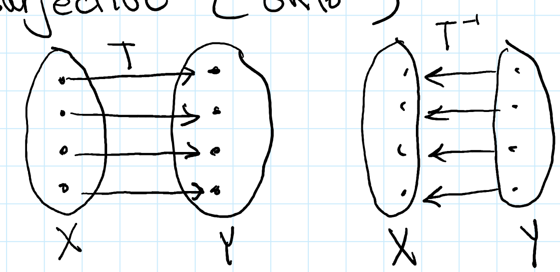
look for review material for Exam 1 by EOD Friday
posted on Brightspace
next Mon will be review
next Wed — Exam 1 will be held during class

§2.4 Inverse of a Linear Transformation

for an inverse to exist, relation must be:

injective (one-to-one) and
surjective (onto)



injective surjective

def'n: function T from $X \Rightarrow Y$ is invertible if $T(x) = y$ has a <u>unique</u> solution $x \in X$ for each $y \in Y$

then $x = T^{-1}(y) \Rightarrow y = T(x)$

also, $T^{-1}(T(x)) = x$ and $T(T^{-1}(y)) = y$ for $\forall x \in X$ and $\forall y \in Y$

finally, if function T is invertible, then so is T^{-1} and then $(T^{-1})^{-1} = T$

ex. in $\mathbb{R} \to \mathbb{R}$:

given $y = \dfrac{x^3 - 1}{5}$ and $x = \sqrt[3]{5y+1}$ is its inverse

similarly, in $\mathbb{R}^n \to \mathbb{R}^n$:

given $\vec{y} = T(\vec{x}) = A\vec{x}$ where A is $n \times n$ matrix
T is a linear transformation

based on previous definition,
the linear transformation above is <u>invertible</u> if $A\vec{x} = \vec{y}$ has a <u>unique</u> solution $\vec{x} \in \mathbb{R}^n$ for $\forall y$ in vector space $\mathbb{R}^n$

i.e., $\text{rref}(A) = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{n \times n} = I_n$ also $\text{rank}(A) = n$

recall: # of leading 1's in rref(A)

if invertible, has one unique solution
if ~~not~~ invertible, has either infinitely many or no sol'ns

special case:

$A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as a solution

zero vector

if A is invertible, it is its only solution
if not, has infinitely many solutions

ex. Is $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$ invertible? yes, because $\text{rref}(A) = I_3$

3x3

ex. find A^{-1} know $\vec{y} = A\vec{x} \rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{array}{cccccc} y_1 & y_2 & y_3 & x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 3 & 8 & 2 \end{array}\right]$$

I_3

$$\left[\begin{array}{ccc|ccc} 10 & -6 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ -7 & 5 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}}_{A^{-1}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}$$ 6R

bringing concepts together:
given an $n \times n$ invertible matrix A
then $AA^{-1} = I_n$ and $A^{-1}A = I_n$

Inverse of a Matrix Product

if $\left.\begin{matrix} A \\ B \end{matrix}\right]$ $n \times n$ invertible matrices

then BA is invertible and

$(BA)^{-1} = A^{-1}B^{-1}$

careful of order!

then it stands to reason that:

given $\left.\begin{matrix} A \\ B \end{matrix}\right]$ $n \times n$ matrices s.t. $BA = I_n$

then A, B are both invertible

i.e., $A^{-1} = B$ and $B^{-1} = A$ and $AB = I_n$

<u>Determinant</u>

Do: given arbitrary $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, compute $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} ad-bc \\ 0 \end{bmatrix}}_{2\times 1} \qquad \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ ad-bc \end{bmatrix}}_{2\times 1}$$

$$\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

becomes

$$ad-bc \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow I_2$$

then, when $ad-bc \neq 0$

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ad-bc) I_2$$

CBW as:

$$\underbrace{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} = I_2$$

$$B \quad A \quad = I_2$$

follow up: when is A invertible? $BA = I_2$

when $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Theorem: 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad-bc \neq 0$

$ad-bc$ is called the determinant of A, denoted as $\det(A)$

i.e., $\det(A) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= ad - bc$

Theorem: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

ex. show $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ it's invertible then find its inverse $= \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\det(A) = ad - bc$
$= 1 - 6 = -5 \neq 0 \quad \therefore \exists A^{-1}$

$A^{-1} = -\frac{1}{5} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ det(A)

$= -\frac{1}{5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$ OR $\begin{bmatrix} -\frac{1}{5} & \frac{3}{5} \\ -\frac{2}{5} & -\frac{1}{5} \end{bmatrix}$

ex. for which value(s) of constant k is $A = \begin{bmatrix} 1-k & 2 \\ 4 & 3-k \end{bmatrix}$ invertible?

$\det(A) = ad - bc$
$= (1-k)(3-k) - 8 \neq 0$ solve for k
$3 - 4k + k^2 - 8 \neq 0$
$k^2 - 4k - 5 \neq 0$
$(k-5)(k+1) \neq 0$
$k \neq 5$ and $k \neq -1$

§3.1 [for Exam 1, NOT ~~only~~ responsible for image, OR ~~not~~ kernel]

HW5 on Gradescope won't be due until Fri pm after Exam 1 however you *SHOULD* make every attempt to complete beforehand since these topics will be on exam

<u>image</u> of a function consists of all values it takes in its largest <u>target space</u>

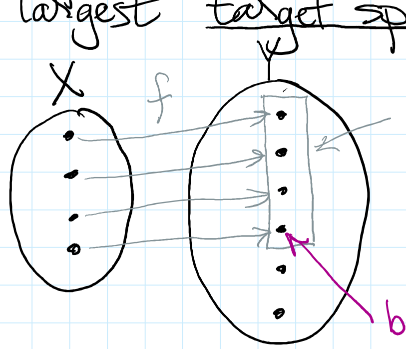


image of f or im(f)

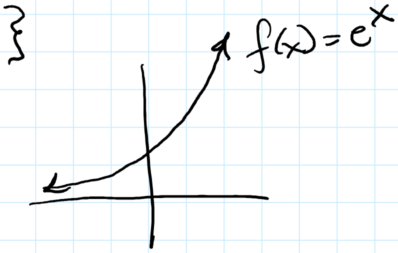
another word for this is range

X: domain of f Y: target space

X: domain of f $\quad$ Y: target space

$\text{im}(f) = \{f(x) : x \in X\}$

$= \{b \text{ in } Y \text{ s.t. } b = f(x) \text{ for some } x \in X\}$

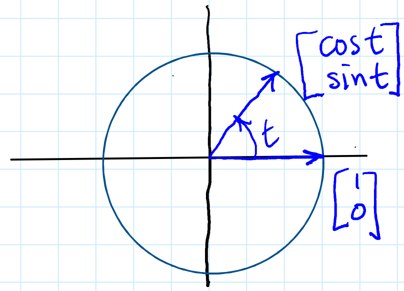


ex. $f(x) = e^x$ $\quad$ $\mathbb{R} \to \mathbb{R}$

domain: all reals $\quad$ image: $\mathbb{R}^+$

target space (perhaps) was all $\mathbb{R}$'s

ex. image of $f(t) = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ $\quad$ $\mathbb{R} \to \mathbb{R}^2$



is the unit circle centered at origin

example: parametrization

e.g., rect $\to$ polar

note: $\underset{\vec{x}}{\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}}$ has a length of 1:

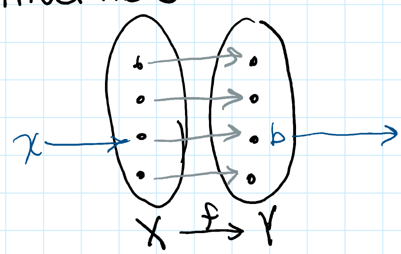
$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$
$$= \sqrt{\Sigma (\text{each component})^2}$$
$$= \sqrt{\cos^2 t + \sin^2 t}$$
$$= 1$$

conclude $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \vec{u}$

unit vector

if $f: X \to Y$ is invertible

then $\text{im}(f) = Y$



then $b = f(x)$

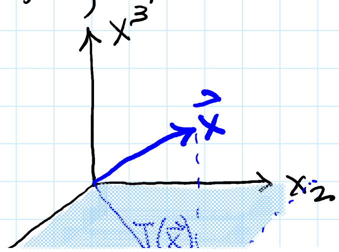
and $x = f^{-1}(b)$

$\therefore$ $b = f(f^{-1}(b))$

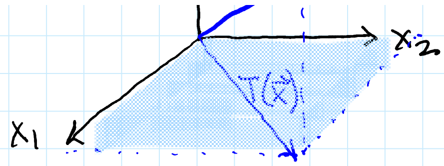
expand def'n:

ex. given linear transformation, T, $\mathbb{R}^3 \to \mathbb{R}^3$ that projects $\vec{x}$ orthogonally onto $x_1 - x_2$ plane

i.e., $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$



$\begin{bmatrix} x_3 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$



then $im(T)$ is x_1-x_2 plane in $\mathbb{R}^3$ consisting of all vectors in form $\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$

Update: image will <u>not</u> be on Exam 1 or HW5